430.714 – Fall 2017

Estimation Theory

Lecture 2

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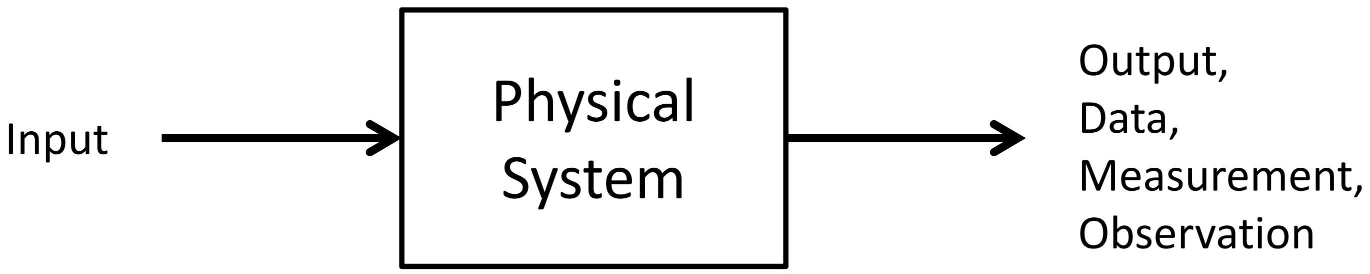
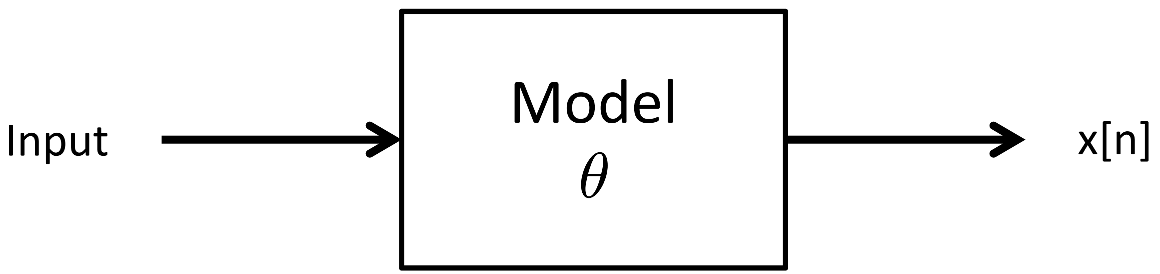
MINIMUM VARIANCE UNBIASED ESTIMATION

Problem Formulation





{ x[0], x[1], …, x[N-1] }

θ: parameter of the model
(latent or hidden variables)

Estimation Problem

Estimation problem (parameter estimation) is to determine θ based on the measurement data.

Estimator: $\hat{\theta} := g(x[0], x[1], \ldots, x[N-1])$

A function of all available data

→ $\hat{\theta}$ is a random variable.

Issues:

- How to choose an estimator?
  - What criteria to use?
- What is the best possible performance?

Bias

Bias of an estimator $\hat{\theta}$:

$$b(\theta) = \mathbb{E}(\hat{\theta}) - \theta$$

$\hat{\theta}$ is an *unbiased* estimator of θ if $\mathbb{E}(\hat{\theta}) = \theta$.
$\hat{\theta}$ is a *biased* estimator of θ if $\mathbb{E}(\hat{\theta}) \neq \theta$.

Example: $x[n] = A + w[n] \qquad w[n] \sim \mathcal{N}(0, \sigma_n^2) \qquad \text{for } n = 0, 1, \ldots, N-1$

- *unbiased*: $\hat{A} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$
- *biased*: $\breve{A} = \frac{1}{2N}\sum_{n=0}^{N-1} x[n]$

$$\mathbb{E}(\breve{A}) = \frac{1}{2}A \begin{array}{ccl} = & A & \text{if } A = 0 \\ \neq & A & \text{if } A \neq 0. \end{array}$$

Bias: $b(\theta) = \mathbb{E}(\hat{\theta}) - \theta$

- Let $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n$ be multiple estimates of θ.
- Let $\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} \hat{\theta}_i$.
- If unbiased and uncorrelated with the same variance,

$$\begin{aligned} \mathbb{E}(\hat{\theta}) &= \theta \\ \mathrm{var}(\hat{\theta}) &= \frac{1}{n^2} \sum_{i=1}^{n} \mathrm{var}(\hat{\theta}_i) = \frac{1}{n} \mathrm{var}(\theta_1). \end{aligned}$$

  Hence, as $n \to \infty$, $\mathrm{var}(\hat{\theta}) \to 0$ and $\hat{\theta} \to \theta$.

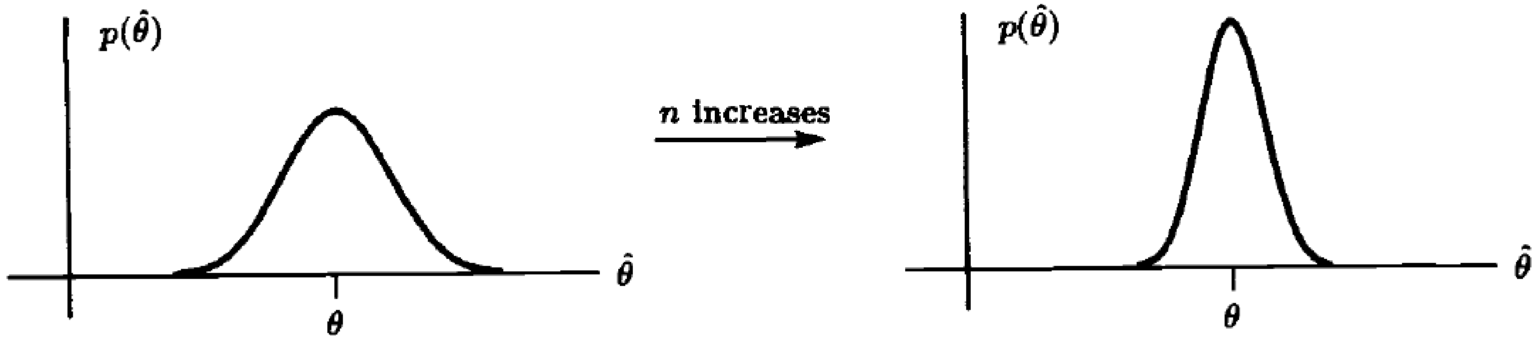
- If biased and $\mathbb{E}(\hat{\theta}_i) = \theta + b(\theta)$,

$$\mathbb{E}(\hat{\theta}) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(\hat{\theta}_i) = \theta + b(\theta).$$

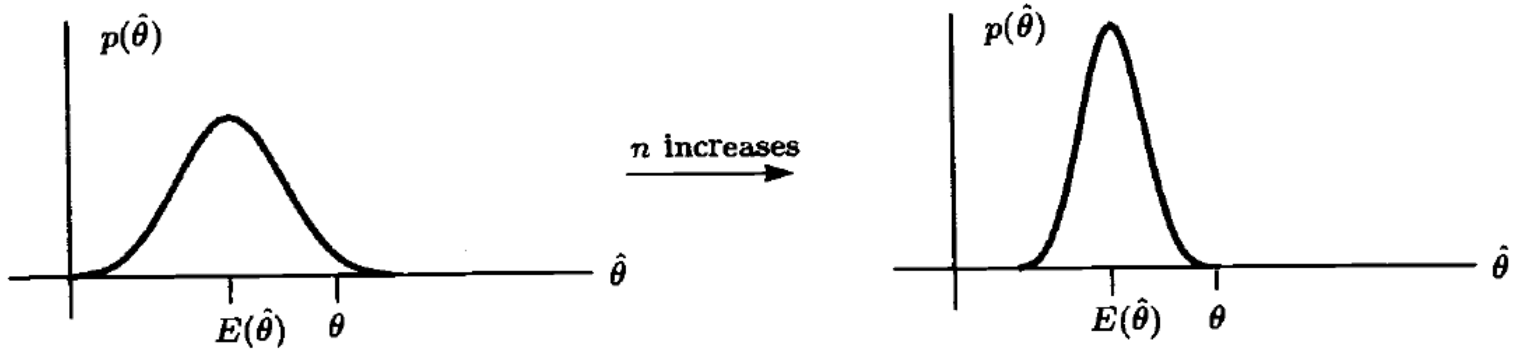
  Hence, as $n \to \infty$, $\hat{\theta} \not\to \theta$.

Bias: $b(\theta) = \mathbb{E}(\hat{\theta}) - \theta$

Unbiased estimator



Biased estimator



Mean Square Error (MSE)

$$\boxed{\text{mse}(\hat{\theta}) = \mathbb{E}\left(\hat{\theta} - \theta\right)^2}$$

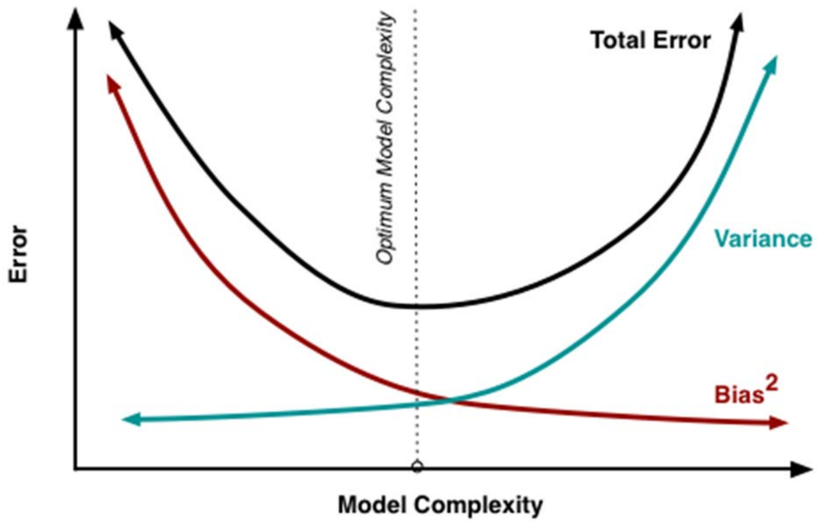
Important: θ is not a random variable.

$$\begin{aligned}
\text{mse}(\hat{\theta}) &= \mathbb{E}\left(\hat{\theta} - \theta\right)^2 \\
&= \mathbb{E}\left[\left(\hat{\theta} - \mathbb{E}(\hat{\theta}) + \mathbb{E}(\hat{\theta}) - \theta\right)^2\right] \\
&= \mathbb{E}\left[\left(\left(\hat{\theta} - \mathbb{E}(\hat{\theta})\right) + \left(\mathbb{E}(\hat{\theta}) - \theta\right)\right)^2\right] \\
&= \mathbb{E}\left(\left(\hat{\theta} - \mathbb{E}(\hat{\theta})\right)^2\right) + \mathbb{E}\left(\left(\mathbb{E}(\hat{\theta}) - \theta\right)^2\right) \\
&= \text{var}(\hat{\theta}) + \left(\mathbb{E}(\hat{\theta}) - \theta\right)^2 \\
&= \text{var}(\hat{\theta}) + b\left(\theta\right)^2
\end{aligned}$$

A function of θ.

We cannot compute $\text{mse}(\hat{\theta})$ without knowing θ in general.
Hence, not a good criterion to use.

Bias-Variance Tradeoff



Minimum Variance Unbiased Estimator

- Find an estimator which minimizes the variance from a set of unbiased estimators.

Cramer-Rao Lower Bound (CRLB)

- – Lower bound of the variance of an unbiased estimator

Rao-Blackwell-Lehmann-Scheffe (RBLS) Theorem

- – Sufficient statistics -> an unbiased estimator