Online Learning to Approach a Person with No-Regret : Supplementary Material

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This document provides the supplementary material to the paper "H. Ahn, Y. Oh, S. Choi, C. J. Tomlin, and S. Oh, *Online Learning to Approach a Person with No-Regret*, IEEE Robotics and Automation Letters (RA-L), submitted." This report includes the detail proof of the Theorem 1 shown in the paper.

1 Proof of Theorem 1

Lemma 1 (Lemma 5.1 in [1])**.** *For $\delta \in (0,1)$, if $\beta_k = 2\log(|\mathcal{Q}|\pi_k/\delta)$, where $\sum \pi_k^{-1} = 1$ and $\pi_k = \pi^2 k^2/6$,*

$$|\mathcal{P}(\mathbf{q}) - \mu_{k-1}(\mathbf{q})| \leq \beta_k^{1/2}\sigma_{k-1}(\mathbf{q})$$

$\forall \mathbf{q} \in \mathcal{Q}$, with probability $1-\delta$.

Lemma 2. *If $|\mathcal{P}(\mathbf{q}) - \mu_{k-1}(\mathbf{q})| \leq \beta_k^{1/2}\sigma_{k-1}(\mathbf{q})\ \forall \mathbf{q} \in \mathcal{Q}$,*

$$r_k \leq \sum_{t=1}^{T_k} 2\beta_k^{1/2}\sigma_{k-1}(\xi_k(t)),$$

where $T_k = |\xi_k|$.

Proof. For ξ_k chosen at the kth round, the GP-UCB algorithm is applied such that:

$$\xi_k = \arg\max_{\xi \in \Xi} \sum_{t=1}^{|\xi|} \left(\mu_{k-1}(\xi(t)) + \beta_k^{\frac{1}{2}}\sigma_{k-1}(\xi(t))\right).$$

Therefore, it is clear that

$$\sum_{t=1}^{T_k} \left(\mu_{k-1}(\xi_k(t)) + \beta_k^{1/2}\sigma_{k-1}(\xi_k(t))\right) \geq \sum_{t=1}^{T^*} \left(\mu_{k-1}(\xi^*(t)) + \beta_k^{1/2}\sigma_{k-1}(\xi^*(t))\right) \geq f(\xi^*)$$

Hence, we have

$$\begin{aligned} r_k &= f(\xi^*) - f(\xi_k) \\ &\leq \sum_{t=1}^{T_k} \left(\mu_{k-1}(\xi_k(t)) + \beta_t^{1/2}\sigma_{k-1}(\xi_k(t))\right) - f(\xi_k) \\ &\leq \sum_{t=1}^{T_k} \left(\mu_{k-1}(\xi_k(t)) - \mathcal{P}(\xi_k(t))\right) + \beta_t^{1/2}\sigma_{k-1}(\xi_k(t)) \\ &\leq \sum_{t=1}^{T_k} 2\beta_k^{1/2}\sigma_{k-1}(\xi_k(t)) \end{aligned}$$

□

Let $\Xi_k \in \Xi$ be a set of all k-combinations in Ξ. For $A \in \Xi_k$, we define $\mathbf{q}(A) = \cup_{\xi \in A} \cup_{t=1}^{|\xi|} \xi(t)$, the set of all states of all paths in A.

Lemma 3. *For $\delta \in (0,1)$ and β_k defined as in Lemma 1, with probability at least $1-\delta$,*

$$\sum_{k=1}^{K} r_k^2 \leq C_1 \beta_K \gamma_K \tag{1}$$

where $C_1 = 8T_{\max}/\log(1+\sigma_\epsilon^{-2})$ and $\gamma_K = \max_{A \in \Xi_K} \mathbb{I}(p_{\mathbf{q}(A)}; \mathcal{P}_{\mathbf{q}(A)})$ is the maximum information gain after K rounds. Here, $\mathcal{P}_{\mathbf{q}(A)}$ and $p_{\mathbf{q}(A)}$ are sets of comfort scores and corresponding observations at states in A, respectively.

Proof. From Lemma 2, we have

$$\begin{aligned} r_k^2 &\leq \left(\sum_{t=1}^{T_k} 2\beta_k^{1/2}\sigma_{k-1}(\xi_k(t))\right)^2 \\ &\leq 4\beta_K \left(\sum_{t=1}^{T_k} \sigma_{k-1}(\xi_k(t))\right)^2 \leq 4\beta_K T_k \sum_{t=1}^{T_k} \sigma_{k-1}^2(\xi_k(t)) \end{aligned}$$

since β_k is nondecreasing. The last inequality is due to the Cauchy-Schwarz inequality. By defining $C_2 = \sigma_\epsilon^{-2}/\log(1+\sigma_\epsilon^{-2}) \geq 1$ as done in [1], we have

$$\begin{aligned} r_k^2 &\leq 4\beta_K T_k \sigma_\epsilon^2 \sum_{t=1}^{T_k} \sigma_\epsilon^{-2}\sigma_{k-1}^2(\xi_k(t)) \\ &\leq 4\beta_K T_k \sigma_\epsilon^2 \left(\sum_{t=1}^{T_k} C_2 \log(1+\sigma_\epsilon^{-2}\sigma_{k-1}^2(\xi_k(t)))\right) \\ &= 8\sigma_\epsilon^2 C_2 T_k \beta_K \left(\frac{1}{2}\sum_{t=1}^{T_k} \log(1+\sigma_\epsilon^{-2}\sigma_{k-1}^2(\xi_k(t)))\right) \end{aligned}$$

Using Lemma 5.3 in [1], for $A_k \in \Xi_k$, we have

$$\mathbb{I}(p_{\mathbf{q}(A_k)}; \mathcal{P}_{\mathbf{q}(A_k)}) = \sum_{\xi \in A_k} \left(\frac{1}{2} \sum_{t=1}^{|\xi|} \log(1 + \sigma_\epsilon^{-2} \sigma_{k-1}^2(\xi(t))) \right)$$

Noting that $|A_k| = k$, we arrive at

$$\sum_{k=1}^{K} r_k^2 \leq 8\sigma_\epsilon^2 C_2 T_{\max} \beta_K \mathbb{I}\big(p_{\mathbf{q}(A_K)}; \mathcal{P}_{\mathbf{q}(A_K)}\big) \leq C_1 \beta_K \gamma_K$$

Lastly, C_1 can be simplified to $C_1 = 8T_{\max}/\log(1 + \sigma_\epsilon^{-2})$ □

Since $R_K^2 \leq K \sum_{k=1}^{K} r_k^2$ using the Cauchy-Schwarz inequality, Theorem 1 has been proven.

References

[1] N. Srinivas, A. Krause, S. M. Kakade, and M. Seeger, "Gaussian process optimization in the bandit setting: No regret and experimental design," in *Proc. of the International Conference on Machine Learning*, 2009.